Preprint typeset in JHEP style. - HYPER VERSION

hep-th/0007217

Noncommutative Tachyonic Solitons. Interaction with Gauge Field*

Corneliu Sochichiu

The Abdus Salam ICTP, strada Costiera 11, 34100 Trieste, Italy
Institutul de Fizică Aplicată AŞ, str. Academiei, nr. 5, Chişinău, MD2028 MOLDOVA
and
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Reg., RUSSIA
E-mail: `sochichi@thsun1.jinr.ru`

Abstract: We show that in the presence of U(1) non-commutative YM interaction the noncommutative tachyonic system exhibits solitonic solutions (even) for finite value of non commutativity parameter θ.

Keywords: Noncommutative geometry, tachyon, Matrix theory.

*Work supported by RFBR grant # 99-01-00190, INTAS grant # 950681, and Scientific School support grant 96-15-0628

Contents

1. Introduction

The recent interest to the non commutative models have its source in their deep relation to the string theory, [1, 2, 3].

For example, the decaying modes of non-BPS unstable branes in IIA and IIB string theories can be described in terms of noncommutative tachyonic solitons [4, 5, 6]. This description is based on solutions found in Ref. [4], when the non-commutativity parameter θ goes to infinity. In this limit one can neglect the kinetic term in the equations of motion, and this allows one obtaining nontrivial solutions depending only on the minima of the tachyonic potential. These solutions are localised in a finite region of the space and can be interpreted as solitons [4]. So far, the tachyonic system was considered on the noncommutative space with constant (and large) noncommutativity parameter θ. This θ corresponds to constant B-field on the brane. In order to get the desired value of θ, one has to take the limit, when B-field is much bigger than metric but small enough to give the large value of θ, [6].

From the other hand one can go beyond the approximation of constant θ (or B-field) and allow also dynamics for this parameter. We put forward the idea to describe the *dynamical* noncommutativity through the *U(1) noncommutative Yang–Mills model* arising from IKKT matrix model [7], at $N = \infty$, [8, 9, 10, 11].

In this model the noncommutativity parameter $\theta = B^{-1}$ arises as the r.h.s of the commutator of the solution $\mathcal{A}_\mu^{(0)} = p_\mu$ to the equations of motion:

$$[p_\mu, p_\nu] = \mathrm{i}B_{\mu\nu}. \tag{1.1}$$

This solution can be seen as one generating a flat noncommutative space-time in Connes' approach [12]. The generic configuration of noncommutative gauge fields

$\mathcal{A}_\mu$ can be described as noncommutative functions on $x^\nu = \theta^{\mu\nu} p_\nu$, which are subject to the Moyal product (or star product) defined as follows,

$$\mathcal{A} * \mathcal{B}(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \partial'_\nu} \mathcal{A}(x)\mathcal{B}(x')\big|_{x'=x}, \tag{1.2}$$

where the ∂_μ and ∂'_μ denote derivatives with respect to x and x' respectively. In Ref. [11] we have shown that in the case of noncommutative gauge field the algebra (1.2) is equivalent to its two dimensional reduction.

Extending the above case one can identify the dynamical noncommutativity parameter with the strength tensor of the gauge field $\mathcal{A}_\mu$.

In what follows, we are going to show that allowing the noncommutative parameter to be dynamical and not just constant one makes possible finding the solitonic solutions without taking any limit.

The plan of the paper is as follows. First, we introduce the model describing the tachyonic system interacting with the gauge field $\mathcal{A}_\mu$. The interaction is introduced as the gauging noncommutative U(1) gauge symmetry from "nothing". After that we consider a class of solitonic ansätze for equations of motion and find the consistent one. Finally, we discuss the results.

2. The model

In Ref. [5], it was claimed that in the presence of B-field an unstable Dp-brane is described by the following noncommutative tachyonic action,

$$S = \int d^{p+1}x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi - V(*\phi) \right), \tag{2.1}$$

where the star reminds that all the products are taken to be star ones, and $V(*\phi)$ is the tachyonic potential [13, 14, 15]. Generally, the particular form of $V(*\phi)$ is not known, except the fact that it has negative slope at the origin (this is why it is tachyonic) and nontrivial minima outside origin, where it is negative. Since the negative energy of the tachyonic potential can compensate the positive brane tension this allows the decaying of branes to nothing [5, 6].

The action (2.1) can be considered as one possessing "invariance" under global gauge symmetry given by constant noncommutative unitary transformations,

$$\phi \to U^\dagger * \phi * U, \qquad U^\dagger * U = 1 \tag{2.2}$$

since $U = \mathrm{e}^{\mathrm{i}\varphi} = \mathrm{const}$, the star products in the above equation are equivalent to ordinary products.

The fields are represented as operators on a Hilbert space $\mathcal{H}$, and symmetry (2.2), just represents the phase invariance of quantum states. Although, the action of this symmetry on fields ϕ is trivial, ($U^\dagger \phi U \equiv \phi$), due to noncommutativity one gets non

trivial gauging of this action. Indeed, non constant U is the picture changing unitary operator which, generally, do not leave ϕ invariant. The action (2.1), is not invariant with respect to this transformation of the field ϕ. The invariance of the action, however, can be restored by the gauging derivatives in (2.1). As in conventional theory, the gauging is obtained by substitution of ordinary derivatives by covariant ones, the only difference being that the gauge fields are also noncommutative,

$$\partial_\mu\phi \to \nabla_\mu\phi \equiv \partial_\mu\phi + \mathrm{i}[\mathcal{A}_\mu, \phi], \tag{2.3}$$

we remind that the commutator in (2.3) is computed using the star product (1.2).

Introducing also the Yang–Mills part for the field $\mathcal{A}$, one has for the total action,

$$S = \int d^{p+1}x \left(\frac{1}{2}\nabla_\mu\phi\nabla_\mu\phi - V(*\phi) - \frac{1}{4g^2}\mathcal{F}^2_{\mu\nu}\right), \tag{2.4}$$

where $\mathcal{F}_{\mu\nu}$ is the gauge field strength,

$$\mathcal{F}_{\mu\nu} = \mathrm{i}\left(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + \mathrm{i}[\mathcal{A}_\mu, \mathcal{A}_\nu]\right), \tag{2.5}$$

here, as well as in the consequent equations all the products are the star ones, note also that the gauge field $\mathcal{A}_\mu$ is in the Hermitian form while the gauge strength $\mathcal{F}_{\mu\nu}$ is an anti-Hermitian one.

Due to the noncommutativity one can eliminate the derivatives from the kinetic terms of action (2.4) by shifting the gauge field $\mathcal{A}_\mu$ as follows,

$$\mathcal{A}_\mu \to p_\mu + \mathcal{A}_\mu, \tag{2.6}$$

where $p_\mu = \theta^{-1}_{\mu\nu}x^\nu \equiv B_{\mu\nu}x^\nu$.

Indeed, the action (2.4) depends on $\mathcal{A}$ only through the covariant derivatives, while the covariant derivative of an arbitrary function f can be represented as,

$$\nabla_\mu f = \partial_\mu + \mathrm{i}[\mathcal{A}, f] = \mathrm{i}[(p_\mu + \mathcal{A}_\mu), f], \tag{2.7}$$

from which it immediately follows that all covariant derivatives in action (2.4) are replaced by commutators $[\mathcal{A}_\mu, f]$.

As a result the action (2.4) looks as follows,

$$S = \int d^{p+1}x \left(\frac{1}{2}[\mathcal{A}_\mu, \phi]^2 - V(\phi) - \frac{1}{4}[\mathcal{A}_\mu, \mathcal{A}_\nu]^2\right), \tag{2.8}$$

where we also put $g = 1$.

Action (2.8) produce the following equations of motion,

$$\frac{\partial V(\phi)}{\partial\phi} - [\mathcal{A}_\mu, [\mathcal{A}_\mu, \phi]] = 0, \tag{2.9}$$

$$[\mathcal{A}_\mu, [\mathcal{A}_\mu, \mathcal{A}_\nu]] + [\phi, [\mathcal{A}_\mu, \phi]] = 0. \tag{2.10}$$

An interpretation which can be given to the action (2.8), is one of a tachyonic field ϕ living on a noncommutative space-time generated by operators $\mathcal{A}_\mu$, when one interpret them in the sense of Connes approach as the (noncommutative) space-time position operators [9]. Indeed, in the case when gauge fields $\mathcal{A}_\mu$ form an irreducible set (i.e. the only function commuting with all $\mathcal{A}_\mu$ is the constant one), one can express the fields in terms of operator functions $\phi = \phi(\mathcal{A})$.

3. Solitonic solutions

Consider the equations of motion (2.9,2.10), and let us look for the static solutions, i.e. ones commuting with $\mathcal{A}_0$,

$$[\mathcal{A}_0, \mathcal{A}_i] = [\mathcal{A}_0, \phi] = 0, \qquad i = 1, \ldots, p. \tag{3.1}$$

This truncates the equation of motion to the following form,

$$\frac{\partial V(\phi)}{\partial \phi} - [\mathcal{A}_i, [\mathcal{A}_i, \phi]] = 0, \tag{3.2}$$

$$[\mathcal{A}_i, [\mathcal{A}_i, \mathcal{A}_j]] + [\phi, [\mathcal{A}_j, \phi]] = 0, \tag{3.3}$$

where Latin indices run through space-like directions of the brane $i = 1, \ldots, p$.

Let us find solutions to eqs. (3.2,3.3). In the case if there were no commutator term in equation (3.2), the solution to this equation would be given by a finite sum [4],

$$\phi = \sum_I a_I \Phi_I, \tag{3.4}$$

where a_I are the minima of the tachyonic potential $V(a)$, treated as an ordinary (commutative) function, and Φ_I are mutually orthogonal projectors to finite dimensional subspaces of the Hilbert space $\mathcal{H}$,

$$\Phi_I \Phi_J = \delta_{IJ} \Phi_J, \tag{3.5}$$

where no sum is assumed over repeated J in the above equation.

Let us return back to the truncated equations of motion (3.2,3.3). The above arguments apply also to our case if the term with double commutator of $\mathcal{A}_i$ with ϕ vanishes, i.e.,

$$[\mathcal{A}_i, [\mathcal{A}_i, \phi]] = 0. \tag{3.6}$$

The equation (3.6), is the Laplace equation in the presence of the gauge field $\mathcal{A}_\mu$.

In what follows our strategy will consist in solving the simple tachyonic equation

$$\frac{\partial V(\phi)}{\partial \phi} = 0, \tag{3.7}$$

and after that finding the gauge field backgrounds which satisfy the remaining equations of motion common with the condition (3.6). We are not going to find the general solution to (3.6), but instead propose a number of possible ansätze, without claiming to enumerate all the possibilities. The ansätze are as follows,

$$
\begin{aligned}
&\text{i)} &\quad [\mathcal{A}_i, \phi_0] &= 0,\\
&\text{ii)} &\quad [\mathcal{A}_i, \phi_0] &= c_i,\\
&\text{iii)} &\quad [\mathcal{A}_i, \phi_0] &= c\mathcal{A}_i,\\
&\text{iv)} &\quad [\mathcal{A}_i, \phi_0] &= \pi_i\phi_0,
\end{aligned}
$$

where ϕ_0 is the solitonic solution to eq. (3.2), and c_i, c and π_i are arbitrary constants.

It is worthwhile to note that in the last case iv) for $\pi_i \neq 0$ the eq. (3.7) is not satisfied but reduces to the form,

$$\frac{\partial \widetilde{V}}{\partial \phi} = 0, \tag{3.8}$$

where the modified tachyonic potential $\tilde{V}$ is given by,

$$\widetilde{V} = V + \frac{1}{2}\pi^2\phi. \tag{3.9}$$

In this case ϕ_0 is a solution to (3.8), given by (3.4), where a_I should be substituted by $\tilde{a}_I$ which now are minima of $\tilde{V}$.

As it is not difficult to show the ansätze iii)-iv) for nonzero c, c_i and π_i lead for $\mathcal{A}_i$ to trivial solutions only. E.g. in the case ii) multiplying from left and right by a factor $(1 - \phi_0)$ gives the following identity (for simplicity we assume that potential $V(a)$ has the only nontrivial minimum),

$$0 = c_i(1 - \phi_0), \tag{3.10}$$

which is satisfied for either trivial ϕ_0 or trivial c_i. In an analogous way one can show that iii) and iv) for nonzero c and π_i are consistent only for trivial $\mathcal{A}_i$, respectively, trivial ϕ_0.

The triviality of $\mathcal{A}_i$ can be interpreted as the space being collapsed to a point.

In what follows consider in more details the remaining case i). In this case the solution for ϕ_0 is given exactly by (3.4). The ansatz consistency condition and the remaining equation of motion for $\mathcal{A}_i$ look as follows,

$$[\mathcal{A}_i, \phi_0] = 0, \tag{3.11}$$

$$[\mathcal{A}_i, [\mathcal{A}_i, \mathcal{A}_j]] = 0. \tag{3.12}$$

Let Ξ_0 denote the finite dimensional subspace of the Hilbert space $\mathcal{H}$, to which ϕ_0 is projecting, $\dim \Xi_0 = N$. And denote the infinite dimensional orthogonal completion to Ξ_0 in $\mathcal{H}$ as $\mathcal{H}_0$. As an operator acting on $\mathcal{H}$, ϕ_0 is the identity one when

restricted to Ξ_0 and vanishes when restricted to $\mathcal{H}_0$. In a convenient basis it can be represented in the following block matrix form,

$$\phi_0 = \begin{pmatrix} \mathbb{I}_0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.13}$$

where $\mathbb{I}_0$ stands for $N \times N$ unity matrix acting on Ξ_0.

Eq. (3.11) implies that gauge field $\mathcal{A}_i$ must have the following block structure,[1]

$$\mathcal{A}_i = \begin{pmatrix} B_i & 0 \\ 0 & C_i \end{pmatrix}, \tag{3.14}$$

where B_i is $N \times N$ hermitian matrix acting on Ξ_0, and C_i is respectively Hermitian operator acting on (infinite dimensional) Hilbert space $\mathcal{H}_0$.

In terms of B_i and C_i the eq. (3.12) is rewritten as, two sets of independent equations,

$$[B_i, [B_i, B_j]] = 0 \tag{3.15}$$

$$[C_i, [C_i, C_j]] = 0 \tag{3.16}$$

The general solution for the finite dimensional part (3.15) is given by a set of commuting matrices [9, 11], while for the eq. (3.16), there are known various solution in different dimensions [16, 17] (see also a recent paper [18]). Let us only note that the one of the simplest solutions is given by operators $C_i^{(0)}$ satisfying,

$$[C_i^{(0)}, C_j^{(0)}] = \mathrm{i}\theta_{ij}, \tag{3.17}$$

for some constant matrix θ_{ij}.

In this case fields $C_i^{(0)}$ are generating a flat noncommutative space corresponding to the "equipotential" surfaces of the solitonic field $\phi_0 = \text{const}$.

4. Conclusions

In this paper we have considered the noncommutative tachyonic field interacting with noncommutative U(1) Yang–Mills model, which is assumed to implement the dynamical noncommutativity.

We have shown that in this case instead of taking the limit of the large noncommutativity parameter θ, (in order to get rid of kinetic term,) one can try find the gauge field background in which the kinetic term vanishes naturally.

We reviewed a number of simple ansätze which solve this condition and have found that the only consistent one is the simplest one in which the solitonic field

[1] Note that from i) it follows that the set of $\mathcal{A}_\mu$ cannot be irreducible, since ϕ_0 commute with all $\mathcal{A}_\mu$ and is not a constant.

is gauge covariant. The existence of the more general ansätze and solutions to the equations of motion, including solitonic ones, also cannot be ruled out. The last point can serve as a topic for future investigations.

Another extension of this work can be seen in the introduction of the supersymmetry. Let us note that the collective coordinate B_i arisen in the decomposition of gauge field $\mathcal{A}_i$ with respect to projector ϕ_0, satisfies the (bosonic part) of the IKKT equations of motion for finite N. These may serve as an indication to a deeper relation of the IKKT model with the dynamics of unstable D-branes.

Since in the $N \to \infty$ limit of eq. (3.15) may possess additional solutions beyond the commutative ones, it would be of interest to analyse also this limit.

Finally, let us note that the presence of the noncommutative U(1) gauge field makes possible the extension of the results of the Ref. [11], concerning the equivalence of the noncommutative models in different dimensions also to the tachyonic system.

Note Added:

When this work was finished the same day appeared a paper [19], containing analogous proposal to introduce non-trivial gauge field interacting with tachyonic field.

Acknowledgments

The author is grateful for the hospitality to the Abdus Salam International Center for Theoretical Physics, where this work has been done. I would like to thank J.H. Schwarz for pointing my attention to the existing problem. I am grateful to Mounia Hssaini for the help in the preparation this publication and also thank her together with M.B. Sedra for valuable discussions.

References

[1] E. Witten, Nucl. Phys. B268 (1986) 253,

[2] N. Seiberg and E. Witten, JHEP 09 (1999) 032, hep-th/9908142,

[3] E. Witten, (2000), hep-th/0006071,

[4] R. Gopakumar, S. Minwalla and A. Strominger, JHEP 05 (2000) 020, hep-th/0003160,

[5] K. Dasgupta, S. Mukhi and G. Rajesh, JHEP 06 (2000) 022, hep-th/0005006,

[6] J.A. Harvey et al., (2000), hep-th/0005031,

[7] N. Ishibashi et al., Nucl. Phys. B498 (1997) 467, hep-th/9612115,

[8] A. Connes, M.R. Douglas and A. Schwarz, JHEP 02 (1998) 003, hep-th/9711162,

[9] C. Sochichiu, JHEP 05 (2000) 026, hep-th/0004062,

[10] C. Sochichiu, Phys. Lett. B485 (2000) 202, hep-th/0005156,

[11] C. Sochichiu, (2000), hep-th/0007127,

[12] A. Connes, (2000), hep-th/0003006,

[13] A. Sen, JHEP 10 (1999) 008, hep-th/9909062,

[14] E.A. Bergshoeff et al., JHEP 05 (2000) 009, hep-th/0003221,

[15] M.R. Garousi, (2000), hep-th/0003122,

[16] N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198 (1998) 689, hep-th/9802068,

[17] H. Aoki et al., Nucl. Phys. B565 (2000) 176, hep-th/9908141,

[18] A. Belhaj et al., (2000), hep-th/0007137,

[19] R. Gopakumar, S. Minwalla and A. Strominger, (2000), hep-th/0003160,